The exponential map for the unitary group SU(2,2)

1994 J. Phys. A: Math. Gen. 27 6799

(http://iopscience.iop.org/0305-4470/27/20/017)

J. Phys. A: Math. Gen. 27 (1994) 6799–6805. Printed in the UK

The exponential map for the unitary group $SU(2,2)$

A O Barut, J R Zeni†§ and A Laufer‡§
Physics Department, University of Colorado, Boulder, CO 80.309–390, USA

Received 21 June 1994

Abstract. In this article, we extend our previous results for the orthogonal group $SO(2,4)$ to its homomorphic group $SU(2,2)$. Here we present a closed finite formula for the exponential of a 4×4 traceless matrix, which can be viewed as the generator (Lie algebra elements) of the $SL(4,C)$ group. We apply this result to the $SU(2,2)$ group, the Lie algebra of which can be represented by Dirac matrices, and discuss how the exponential map for $SU(2,2)$ can be written by means of Dirac matrices.

1. Introduction

In a previous paper (Barut, Zeni and Laufer 1994), the present authors obtained a closed finite formula for the exponential which maps the Lie algebra into the defining representation of orthogonal groups and, in particular, for $SO_+(2,4)$ group. This result is a generalization of the well known and important formulae for the $SO(3)$ group (Barut 1980) and the analogous result for the Lorentz group $SO_+(1,3)$ (Zeni and Rodrigues 1990).

The present article deals with the exponential which maps the Lie algebra into the defining representation of the $SU(2,2)$ group, which is the covering group of the $SO_+(2,4)$ group. The result presented here can be viewed as a generalization of the recent result to the $SL(2,C)$ group (Zeni and Rodrigues 1992)

$$e^F = \cosh z + \hat{F} \sinh z \tag{1.1}$$

where $F = (e_i + \mathrm{i} b_i)\sigma_i$ is a complex vector expanded in the Pauli matrices, the complex variable z is such that $z^2 = F^2$ and $\hat{F} = F/z$. We remark that the above result contains the particular case of the $SU(2)$ group when $e_i = 0$.

The group $SU(2,2)$ and its homomorphic group $SO_+(2,4)$ has several applications in theoretical physics (Barut and Brittin 1971). For instance, it is the largest group that leaves the Maxwell equations invariant (Bateman 1910); from its subgroups $SO(3)$ and $SO(2,1)$, we can obtain the whole spectrum of the hydrogen atom as predicted by Schrödinger theory (Barut 1972); and more recently it has been used in spin-gauge theories in an attempt to generalize the minimal coupling (Barut and McEwan 1984, Chisholm and Farwell 1989, Dehnen and Ghaboussi 1986).

Also, the unitary groups play an important role in quantum mechanics (Barut and Raczcka 1986) and we can find in the literature several articles dealing with the

† Permanent address: Depto Ciencias Naturais, FUNREI, São João del Rei, MG, Brazil, 36.300.
§ Work partially supported by CAPES/Brazil and DAAD/Germany.
‡ Permanent address: Physics Department, University of Konstanz, mailbox 5560 M678, 78434 Konstanz, Germany (e-mail alex@spock.physik.uni-konstanz.de).

0305-4470/94/206799+07$19.50

parametrizations of these groups (see Barnes and Delbourgo (1972) and references therein). For instance, Bincer (1990) presents a parametrization for the exponential through a set of orthonormal vectors that must be computed from the diagonal form of the matrix. We remark that, if one uses our method to obtain the exponential (Barut *et al* 1994), we need only compute the eigenvalues of the matrix; no further computations are involved, such as the eigenvectors.

We remark that, besides its application in group theory, the exponential of a matrix has an important application in the solution of a system of differential equations, as discussed in Barut *et al* (1994), so the method developed in our present and previous papers can be useful for this study. A comprehensive review of methods to exponentiate an arbitrary matrix is given in Moler and van Loan (1978) and references therein.

The plan of the present paper is as follows. In section 2, we obtain the exponential of a 4×4 traceless matrix; in section 3, we discuss some particular cases of that exponential; in section 4, we study the representation of the exponential in the Dirac algebra, in particular, the cases when the generator is either the sum of a vector and an axial vector, or a bivector; in section 5, we make our final comments.

2. The exponential of a 4×4 traceless matrix

The method presented previously by the authors (Barut *et al* 1994) can be straightforwardly generalized to exponentiate any given matrix. Basically, the algorithm presented was based on the Hamilton–Cayley theorem and emphasizes that the exponential must be written by means of the eigenvalues. Also remarkable is the use of the (square-root) discriminant, related to the characteristic equation of the matrix, as a multiplier to simplify the expressions of the coefficients that appear in the recurrence relation. Finally, we analyse the coefficients, examining each eigenvalue separately.

We are going to apply the above algorithm to the generators of $SL(4, C)$. We recall that the Lie algebra of $SL(n, C)$ is defined by

$$sl(n, C) = \{H \in C(n) \text{ such that } \mathrm{e}^{H} \in SL(n, C)\} \tag{2.1}$$

where $C(n)$ is the space of $n \times n$ complex matrices.

Therefore, the generators of $SL(n, C)$ are traceless, since we have $\det \mathrm{e}^{H} = \mathrm{e}^{\mathrm{Tr}\, H}$.

From now on we restrict ourselves to 4×4 matrices.

The characteristic equation of a 4×4 matrix is given by

$$\begin{aligned}\det(H - \lambda I) &= \lambda^4 - d_0\lambda^3 - a_0\lambda^2 - b_0\lambda - c_0 \\ &= (\lambda - w)(\lambda - x)(\lambda - y)(\lambda - z) = 0\end{aligned} \tag{2.2}$$

where w, x, y and z are the eigenvalues.

The matrices representing the generators of $SL(4, C)$ are traceless, so, in this case, the sum of the eigenvalues vanishes

$$d_0 = w + x + y + z = \mathrm{Tr}\, H = 0. \tag{2.3}$$

2.1. The recurrence relations

The Cayley–Hamilton theorem says that a matrix satisfies a matrix equation identical to its characteristic equation and, therefore, we can write all higher powers of H in terms of the first powers (Barut *et al* 1994)

$$H^{(4+i)} = d_i H^3 + a_i H^2 + b_i H + c_i. \tag{2.4}$$

So the series for e^H becomes a series in the coefficients of the above equation. Multiplying the recurrence relation, equation (2.4), by H, and using the Hamilton–Cayley theorem related to equation (2.2), one obtains recurrence relations for the coefficients, which hold for $i \geqslant 0$

$$a_{i+1} = b_i + d_i a_0 \qquad b_{i+1} = c_i + d_i b_0 \qquad c_{i+1} = d_i c_0 \qquad d_{i+1} = a_i. \tag{2.5}$$

From the above relations we can also show that ($i \geqslant 2$)

$$a_{i+2} = a_i a_0 + a_{i-1} b_0 + a_{i-2} c_0. \tag{2.6}$$

In the next step, as outlined in Barut *et al* (1994), we introduce the *square-root of the discriminant* of equation (2.2), indicated hereafter by m:

$$m = (w-x)(w-y)(w-z)(x-y)(x-z)(y-z). \tag{2.7}$$

We write the first coefficients, a_0, b_0 and c_0 by means of the eigenvalues, according to equation (2.2), and from equation (2.6), we find that the general term for the coefficients a_i, multiplied by m, is given by ($i \geqslant 0$)

$$ma_i = t(w,y,z)x^{5+i} + t(x,w,z)y^{5+i} + t(w,x,y)z^{5+i} + t(y,x,z)w^{5+i} \tag{2.8}$$

where we have made use of the *alternating* $t(w,y,z)$ function of three variables

$$t(w,y,z) = (w-y)(y-z)(z-w). \tag{2.9}$$

Now, based on equation (2.8), the series for the coefficients can be summed easily since we can write the other coefficients by means of a_i, according to equation (2.5). For instance

$$b_{i+1} = a_{i-2}c_0 + a_{i-1}b_0. \tag{2.10}$$

In order to write down the exponential, it is convenient to introduce the *symmetric* $s(w,y,z)$ function in three variables and the *product* of the symmetric and alternating functions, indicated hereafter by $st(w,y,z)$

$$s(w,y,z) = wy + wz + yz \qquad st(w,y,z) = s(w,y,z)t(w,y,z). \tag{2.11}$$

2.2. The closed, finite formula for the exponential of a 4 × 4 traceless matrix

$$\begin{aligned} me^H = &-wxyz\left(t(w,y,z)\frac{e^x}{x} + t(x,w,z)\frac{e^y}{y} + t(w,x,y)\frac{e^z}{z} + t(y,x,z)\frac{e^w}{w}\right)1 \\ &+ (st(w,y,z)e^x + st(x,w,z)e^y + st(w,x,y)e^z + st(y,x,z)e^w)H \\ &+ (xt(w,y,z)e^x + yt(x,w,z)e^y + zt(w,x,y)e^z + wt(y,x,z)e^w)H^2 \\ &+ (t(w,y,z)e^x + t(x,w,z)e^y + t(w,x,y)e^z + t(y,x,z)e^w)H^3. \end{aligned} \tag{2.12}$$

3. Some special cases of the exponential

3.1. The case when $b_0 = 0$

Now we are going to see that the above formula for the exponential simplifies considerably in the case when the characteristic equation for H, equation (2.2), has no term in the first power, i.e. $b_0 = 0$. In this case, the recurrence relations, equation (2.4), for the even (odd) powers involves only the even (odd) powers.

If $b_0 = 0$ we have a quadratic equation in the square of the eigenvalue of H, so we can set $w = -x$ and $z = -y$ and work with only two eigenvalues, x and y. The Hamilton–Cayley theorem, related to equation (2.2), becomes

$$H^4 - (x^2 + y^2)H^2 + x^2y^2 = 0. \tag{3.1}$$

The square-root of the discriminant m, equation (2.7), reduces in this case to

$$m = -4xy(x^2 - y^2)^2. \tag{3.2}$$

Therefore, the series for e^H, equation (2.12), in the case when $b_0 = 0$, is given by

$$\begin{aligned} \mathrm{e}^H = {} & \frac{x^2 \cosh y - y^2 \cosh x}{x^2 - y^2} 1 + \frac{\cosh x - \cosh y}{x^2 - y^2} H^2 + \frac{x^3 \sinh y - y^3 \sinh x}{xy(x^2 - y^2)} H \\ & + \frac{y \sinh x - x \sinh y}{xy(x^2 - y^2)} H^3. \end{aligned} \tag{3.3}$$

3.2. Further simplifications: the case when $H^2 = x^2 I$

In this case, the square of the generator can be identified with the square of one eigenvalue, say x, which is a particular situation of the Hamilton–Cayley equation given above, equation (3.1). Examples include the important cases when the generator is either a vector (or axial vector) or a bivector from Dirac algebra. This last case is just the Lorentz group.

Therefore, substituting for $H^2 = x^2$ in equation (3.3), we obtain

$$\mathrm{e}^H = \cosh x + \frac{\sinh x}{x} H. \tag{3.4}$$

This is the formula given in Zeni and Rodrigues (1992) for the exponential of the generators of the Lorentz group. We remark that Zeni and Rodrigues (1992) proved in a very simple way that every proper and orthochronous Lorentz transformation can be written as the exponential of some generator.

4. The $SU(2,2)$ group and Dirac algebra

We recall that the Lie algebra of the $SU(2,2)$ group is defined by Kihlberg *et al* (1966)

$$su(2,2) = \{H \in C(4) \text{ such that } H^\dagger\beta = -\beta H\} \tag{4.1}$$

with $\beta = \mathrm{diag}(1, 1, -1, -1)$.

Also, the matrix algebra $C(4)$ is isomorphic to the Dirac algebra and, therefore, the generators of $SU(2,2)$ can be represented by an appropriate set of Dirac matrices.

The *standard* representation for the Dirac matrices is given by

$$\gamma_0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{4.2}$$

where $i \in [1,3]$ and we set $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ so its square is *negative*, i.e. $\gamma_5^2 = -1$ and it is *skew-Hermitian*.

A general element of the Lie algebra of $SU(2,2)$ can be written as follows

$$H = \mathrm{i}V + F + \gamma_5 A + \mathrm{i}c\gamma_5 = \mathrm{i}v^\mu\gamma_\mu + F^{\mu\nu}\gamma_{\mu\nu} + \gamma_5 a^\mu\gamma_\mu + \mathrm{i}c\gamma_5 \tag{4.3}$$

where $\gamma_{\mu\nu} = \gamma_\mu\gamma_\nu$, $\mu \leqslant \nu$, $\mu,\nu \in [0,3]$. The components v^μ, $F^{\mu\nu}$, a^μ and c are *real* numbers.

4.1. Vector ($H = \mathrm{i}V$) or axial vector ($H = \gamma_5 A$)

In this case, the square of the generator is a *real* number

$$\begin{aligned} &\text{if } H = \mathrm{i}V = \mathrm{i}v^\mu\gamma\mu \Rightarrow H^2 = -V^2 = -v^\mu v_\mu = x^2 \\ &\text{if } H = \gamma_5 A = \gamma_5 a^\mu\gamma_\mu \Rightarrow H^2 = A^2 = a^\mu a_\mu = x^2 \end{aligned} \tag{4.4}$$

the eigenvalues present in equation (3.1) are equal to each other, i.e. we have $x^2 = y^2$.

The previous formula for the exponential of H, equation (3.4), holds in every case, i.e. V can be length-like ($V^2 = 0$), space-like ($V^2 < 0$) or time-like ($V^2 > 0$).

4.2. Bivector ($H = F$)

The square of a bivector in the Dirac algebra can be *formally* identified with a complex number, which is one of the eigenvalues, say x. The other eigenvalue y is related to x by complex conjugation. The imaginary unit is represented by γ_5.

Let us write the bivector as follows

$$F = F^{\mu\nu}\gamma_{\mu\nu} = (e^i + \gamma_5 b^i)\gamma_{i0}. \tag{4.5}$$

In the latter form, it is easy to compute the square of F (Zeni and Rodrigues 1992)

$$F^2 = e^2 - b^2 + 2\gamma_5 e \cdot b. \tag{4.6}$$

From the above equation, we can deduce the explicit form of the Hamilton–Cayley theorem and in this case

$$F^4 - 2(e^2 - b^2)F^2 + 4(e \cdot b)^2 + (e^2 - b^2)^2 = 0. \tag{4.7}$$

If we compare equation (4.7) with equation (3.1), we find that the eigenvalues are given by

$$x^2 = e^2 - b^2 + 2\mathrm{i}e \cdot b \qquad y^2 = e^2 - b^2 - 2\mathrm{i}e \cdot b. \tag{4.8}$$

Observe that the expression for x is the same expression as that for F^2. We have only to replace the imaginary unit for γ_5. Therefore, equation (3.4) applies to the exponential of a bivector.

4.3. The sum of a vector and an axial vector ($H = W = \mathrm{i}V + \gamma_5 A$)

In this case, we are going to show that the fourth power of the generator can be written by means of the second power and the identity and, therefore, equation (3.3) applies for the exponential of a V–A generator. Also, we obtain an explicit expression for the second and third power by means of V and A, equation (4.16), which can further simplify computations with the exponential, equation (3.3).

From now on, we indicate that $W = H$ for the generator, so we have

$$W = \mathrm{i}V + \gamma_5 A = (\mathrm{i}V^\mu + \gamma_5 A^\mu)\gamma_\mu. \tag{4.9}$$

Computing the second and fourth power of W, we find

$$W^2 = A^2 - V^2 + \mathrm{i}\gamma_5(AV - VA) \tag{4.10}$$

$$W^4 = 2(A^2 - V^2)W^2 + 4(A \bullet V)^2 - (A^2 + V^2)^2 \tag{4.11}$$

Therefore, a V–A generator satisfies equation (3.1) and can be exponentiated as in equation (3.3).

Observe that if $V = A$, i.e. $W = (1 + \gamma_5)V$, it implies $W^2 = 0$ and $\mathrm{e}^W = 1 + W$.

We introduce $A \bullet V$ as the *inner product* in the Dirac algebra (also $V^2 = V \bullet V$)

$$V \bullet A = \tfrac{1}{2}(VA + AV) = v^\mu a_\mu.$$

We also remark that $(AV - VA)^2 = 4(A \bullet V)^2 - 4A^2V^2 \stackrel{\text{def}}{=} \Delta$. If we compare equation (4.11) with equation (3.1), we see that the eigenvalues are given by

$$x^2 = A^2 - V^2 + \sqrt{\Delta} \qquad y^2 = A^2 - V^2 - \sqrt{\Delta}. \tag{4.12}$$

To get a convenient expression for the third power of W, we introduce a new element W^*, defined by

$$W^* = A + \mathrm{i}\gamma_5 V. \tag{4.13}$$

The products WW^* and W^*W will be called here *bicomplex numbers*, i.e. we now have two imaginary *commutative* units, γ_5 and i. Moreover, the above products are related to each other through complex conjugation respective to γ_5, i.e. we have

$$\begin{aligned} u &\stackrel{\text{def}}{=} WW^* = 2\mathrm{i}A \bullet V + (A^2 + V^2)\gamma_5 \\ \bar{u} &\stackrel{\text{def}}{=} W^*W = 2\mathrm{i}A \bullet V - (A^2 + V^2)\gamma_5. \end{aligned} \tag{4.14}$$

It is remarkable that the product $u\bar{u} = \bar{u}u$ is a real number, which is just the *determinant* of $H = W$ in the matrix representation (cf equation (4.11) above)

$$u\bar{u} = \bar{u}u = -4(A \bullet V)^2 + (A^2 + V^2)^2. \tag{4.15}$$

Based on equation (4.15), we obtain the inverse of W (when there is an inverse, i.e. $u\bar{u} \neq 0$) as

$$W^{-1} = W^*u^{-1} = \frac{W^*\bar{u}}{u\bar{u}}. \tag{4.16}$$

To verify that the above expression just defines the *bilateral* inverse, we call attention to the fact that γ_5 anticommutes with W, so we have $\bar{u}W = Wu$.

Now considering that $W^3 = W^4W^{-1}$, it follows from equations (4.11) and (4.16) that the third power of W is given (if $\bar{u}u \neq 0$) by

$$W^3 = 2(A^2 - V^2)W - W^*\bar{u} \tag{4.17}$$

which is easily expressed by means of the Dirac matrices through equations (4.9) and (4.13).

5. Conclusions

In this article we presented a finite closed formula for the exponential of a 4×4 traceless matrix, equation (2.12). It can be viewed as the exponential of a generator of the $SL(4, C)$ group, which includes the $SU(2, 2)$ group as a subgroup. Our approach in obtaining the exponential is based on our previous work (Barut *et al* 1994). Equation (2.12) is a generalization of the exponential for generators of the $SL(2, C)$ group presented in Zeni and Rodrigues (1992).

The finite formula for the exponential, equation (2.12), involves only the computations of the eigenvalues and the first three powers of the matrix, no further computations (e.g. eigenvectors) are needed to obtain the exponential (cf Moler and van Loan 1978).

We have also presented some special cases of this exponential, equations (3.3) and (3.4). They include the important cases when the generator of the $SU(2, 2)$ group is identified either with a bivector or the sum of a vector and an axial vector, as discussed in section 4. For both cases, we give explicit expressions for the eigenvalues of the generators, equations (4.8) and (4.12), as derived from Dirac algebra. Moreover, in the case of a V–A generator, we obtained a simple expression for the third power of the generator by means of V and A, equation (4.17), which is needed to exponentiate the generator (see equation (3.3)).

Acknowledgments

The authors are thankful to Professor W A Rodrigues Jr, P Lounesto and H Dehnen for several discussions on the subject. JRZ and AL are thankful to Professor A O Barut for his kind hospitality in Boulder. AL and JRZ are also thankful to DAAD/Germany and CAPES/Brazil, respectively, for the fellowships that supported their stay in Boulder.

References

Barnes K J and Delbourgo R 1972 *J. Phys. A: Math. Gen.* **5** 1043–53

Barut A O 1964 *Phys. Rev.* B **135** 839–42

—— 1972 *Dynamical Groups and Generalized Symmetries in Quantum Theory* (Christchurch: University of Canterbury Press)

—— 1980 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Dover)

Barut A O and Brittin W E (ed) 1971 *Lectures on Theoretical Physics: De Sitter and Conformal Groups* vol XIII (Boulder, CO: Colorado University Press)

Barut A O and McEwan J 1984 *Phys. Lett.* **135** 172–4

Barut A O and Raczka R 1986 *Theory of Group Representation and Applications* 2nd edn (Singapore: World Scientific)

Barut A O, Zeni J R and Laufer A 1994 The exponential map for the conformal group $O(2, 4)$ *J. Phys. A: Math. Gen.* **27** 5239–50

Bateman H 1910 *Proc. London Math. Soc.* **8** 288

Bince A M 1990 *J. Math. Phys.* **31** 563–67

Chisholm J S R and Farwell R S 1989 *J. Phys. A: Math. Gen.* **22** 1059–71

Dehnen H and Ghaboussi F 1986 *Phys. Rev.* D **33** 2205–11

Kilhberg A, Müller V F and Halbwachs F 1966 *Commun. Math. Phys.* **3** 194–217

Moler C and van Loan C 1978 *SIAM Rev.* **20** 801–36

Zeni J R and Rodrigues W A 1990 *Hadronic J.* **13** 317–23

—— 1992 *Int. J. Mod. Phys.* A **7** 1793–817